CMPUT 466
Machine Learning: Day 2

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Winter 2024

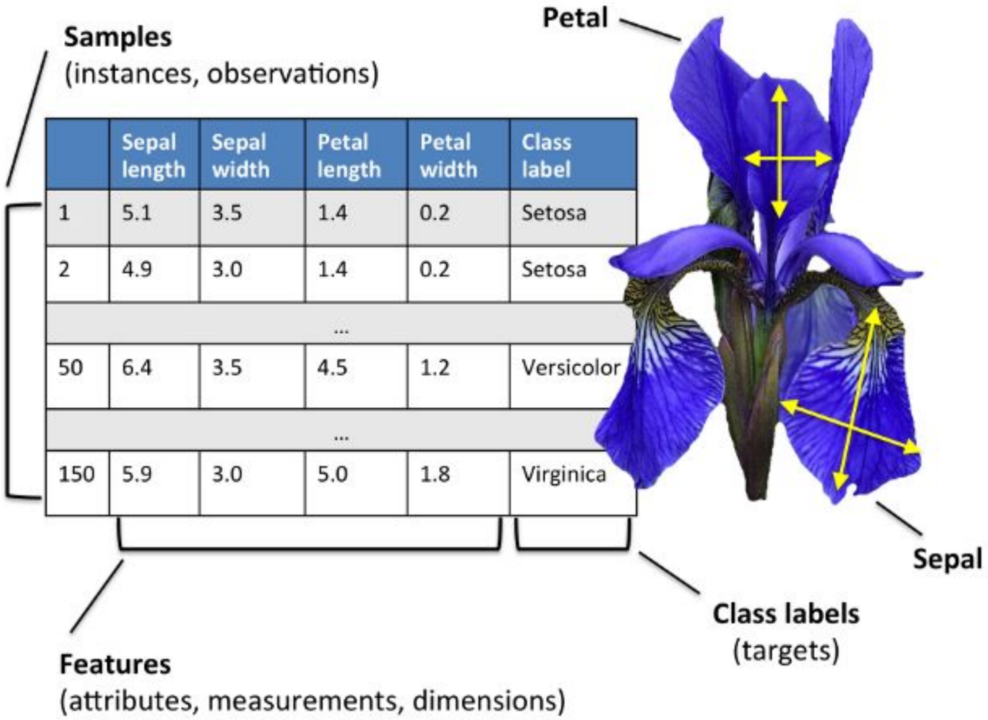
Many of these slides are derived from Alona Fyshe, Alex Thomo. Thanks!

More resources…

- From the TAs (thank you TAs)
  - Shalev-Shwartz, S., & Ben-David, S. (2014). Understanding machine learning: From theory to algorithms. Cambridge university press.
  - Bach F. (2023). Learning Theory from First Principle. (https://www.di.ens.fr/~fbach/ltfp_book.pdf)

Representing data for ML?

Data for ML: A Dataset of a Flower



Samples
(instances, observations)

	Sepal length	Sepal width	Petal length	Petal width	Class label
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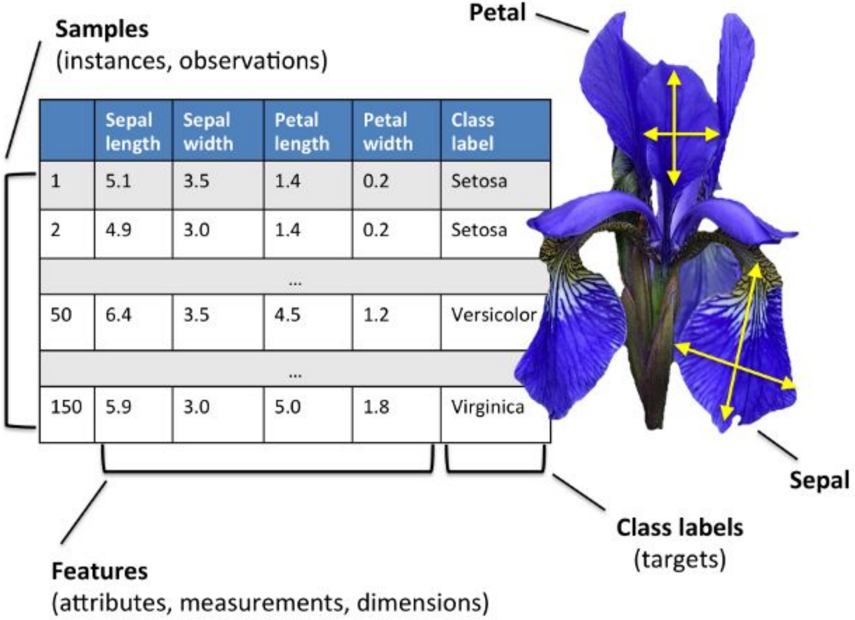
...					
150	5.9	3.0	5.0	1.8	Virginica

Features
(attributes, measurements, dimensions)

Class labels
(targets)

Iris Dataset

- Four **features**, plus the **class label**
-



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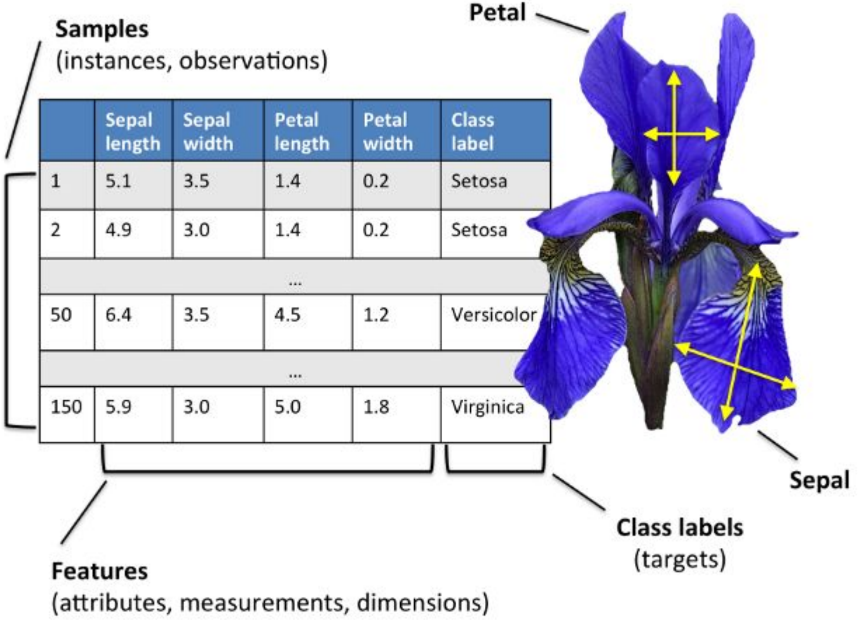
https://medium.com/analytics-vidhya/exploratory-data-analysis-uni-variate-analysis-of-iris-data-set-690c87a5cd40

Iris Dataset

- Four features, plus the class label
- Our **task** is to predict class label (flower type) from the 4 features



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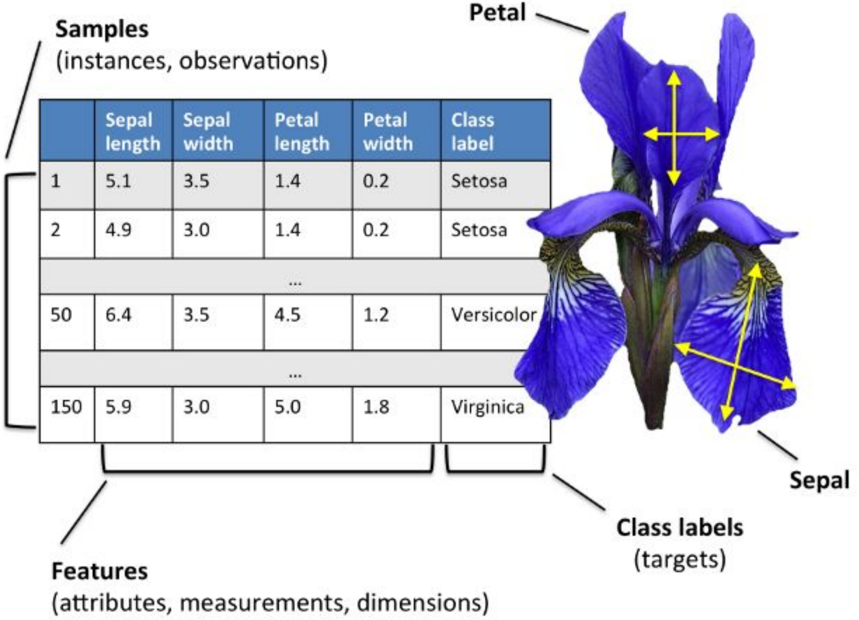
Iris Dataset

- Four features, plus the class label
- Our task is to predict class label (flower type) from the 4 features
- To **graph these feature vectors**, we would need a **4D space**
    - Difficult to visualize



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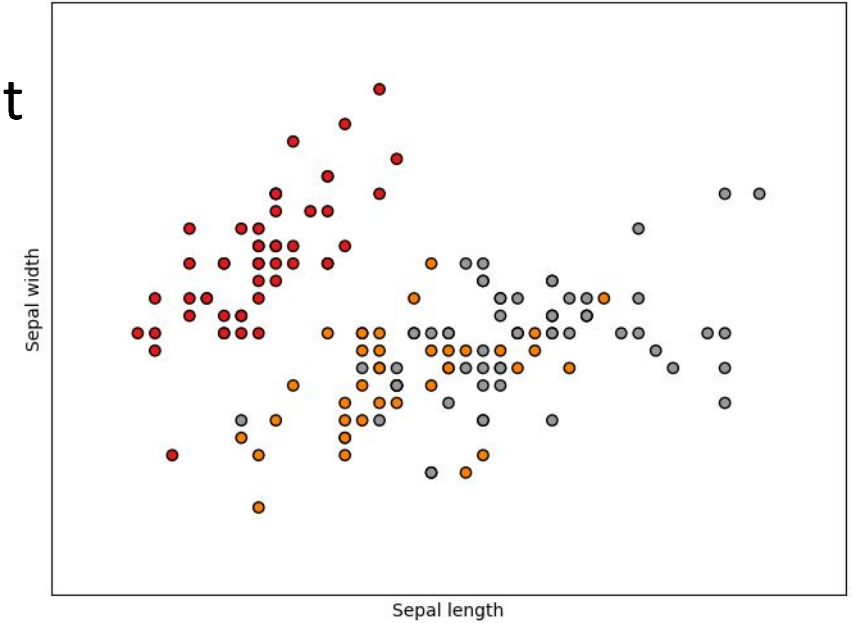
Dimensions as Features

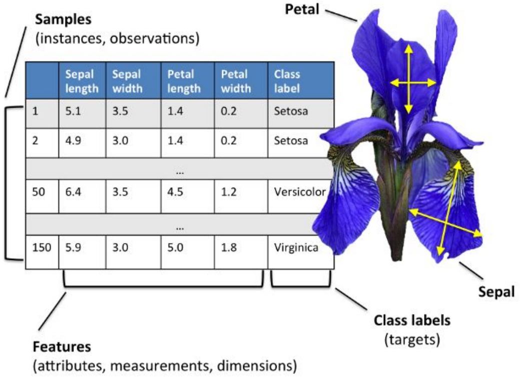
- We can use the **dimensions of a vector** to represent the values for different features in our data
  - E.g. the very famous Iris dataset

- In the figure →
  - X: sepal length
  - Y: sepal width
  - Color of dot: flower type



Iris Dataset

- Each of *the 4 features* are ***continuous***
-



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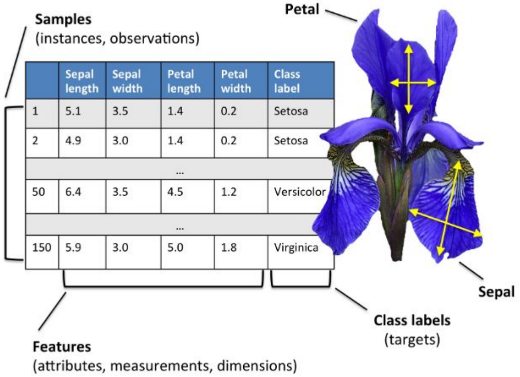
Iris Dataset

- Each of the 4 features are *continuous*
- The ***Class label*** is ***discrete***



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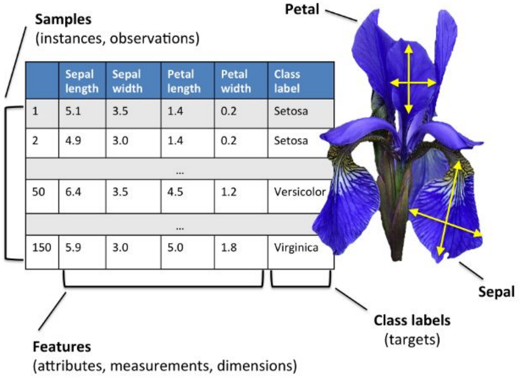
- The Class label is *discrete*

- How to represent **class label**?



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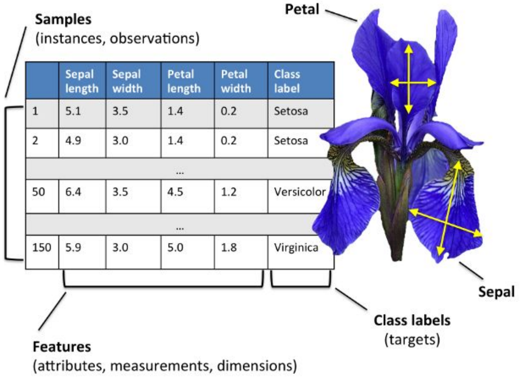

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- Each of the 4 features are *continuous*
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- How to represent **class label**?
- Unique integer values
  - (e.g. 1=Setosa, 2=Versicolor, 3=Virginica)
- One hot vector
  - [1, 0, 0] -> Setosa
  - [0, 1, 0] -> Versicolor
  - [0, 0, 1] -> Virginica

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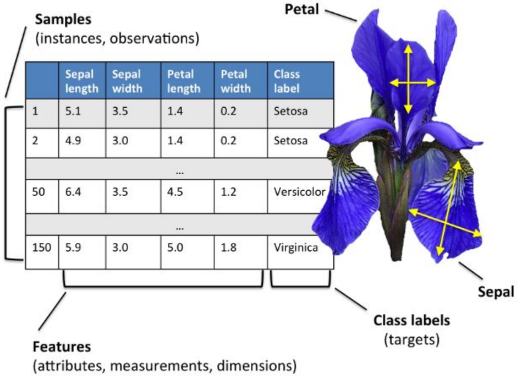

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Some terminology/notation…

Discrete features

- **Class label** is an example of **a discrete feature**
    - As opposed to continuous features like length and width
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- **Features** can also be **discrete**
  - E.g. number of petals
  - Favorite movie
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Discrete features

- Class label is an example of a discrete feature
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- Features can also be discrete
  - E.g. number of petals
  - Favorite movie
- Sometimes these **features** are **ordinal** (they have an **ordering**)
  - Number of petals
  - Not favorite movie

Discrete features for ML

- When features are **ordinal**, it can make sense to represent them with **integer numbers**
- When features are **categorical** (i.e. non-ordinal) **one hot vectors** work better

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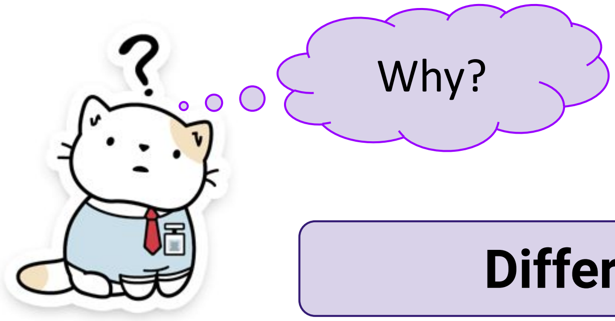

Different Meaning: Ordinal is a relationship

More Terms/Notation

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  - The number of dimensions is the **length** of the list
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- A vector is a list of numbers
  - The number of dimensions is the length of the list
- A matrix is a table of numbers, so it has a **length** and a **height**
  - E.g. 5x2, 10x100
  - Convention is **R**ows x **C**olumns (e.g., **R**oman **C**atholic, **R**o**c**k, **R**oll **C**all, **R**ate **C**lass)
-

More Terms/Notation

- A vector is a list of numbers
  - The number of dimensions is the length of the list
- A matrix is a table of numbers, so it has a length and a height
  - E.g. 5x2, 10x100
  - Convention is **R**ows x **C**olumns
- By this same logic, a **vector is** actually **a matrix** with **length or height** of **1**
  - 6x1 is a column vector with 6 elements
  - 1x3 is a row vector with 3 elements

Notational Conventions

- Square brackets to denote boundaries of vectors/matrices

- Convention is for variable names that denote vectors to be
  - Lowercase **a**
  - Bold or have an arrow over them (not always adhered to if the context makes the form of the variable clear) $\vec{\mathbf{a}}$

- Matrices
  - Uppercase
  - Plain font A

Linear Algebra Notational Recall

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- Communicate the size of a vector like this: $A \in \mathbb{R}^{n \times p}$

- Transpose (T) means to swap rows for columns $A^T \in \mathbb{R}^{p \times n}$

Example: text documents

- Representing text as a feature vector
- Example (nonsensical) text:
    - D1: **brown cat brown cat dog cat mouse**
    - D2: **brown cat mouse mouse mouse**
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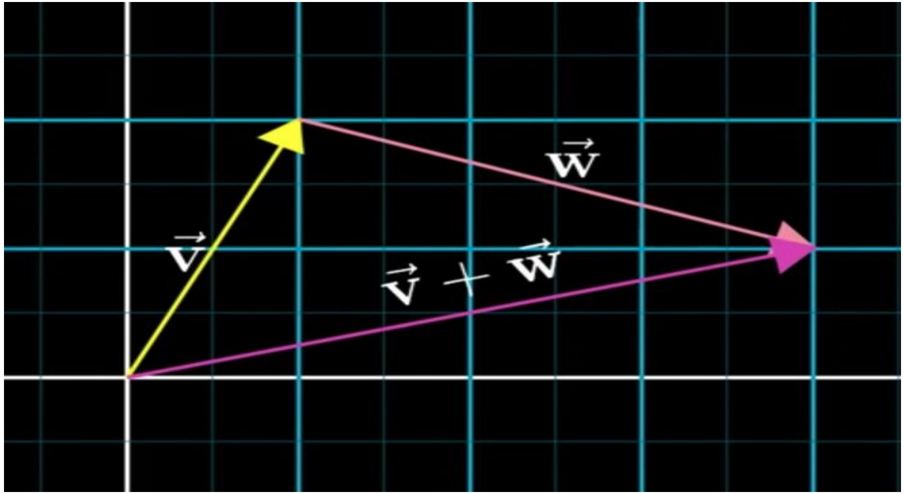
- Identify vocabulary (all words across all documents)
  - **brown**, **cat**, **dog**, **mouse**, **meow** (this is the feature order below)
- Features are the # of occurrences of each vocabulary word in doc.
  - D1: [2, 3, 1, 1, 0]
  - D2: [1, 1, 0, 3, 0]
  - D3: [2, 1, 1, 0, 1]

Vector Addition



```
  v: [1,  2]
  w: [3, -1]


v+w: [4,  1]
```

Pic from youtube playlist video 1

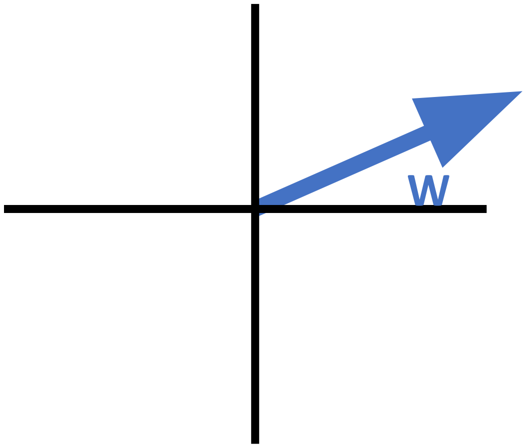
Vector addition for our text dataset?

- Recall:
  - D1: [**2**, **3**, **1**, **1**, **0**]
  - D2: [**1**, **1**, **0**, **3**, **0**]

- What does it mean to have a new document A = D1 + D2?

- I.e. what document would give us a vector equivalent to A = D1 + D2?

Scalar multiplication for vectors



3w

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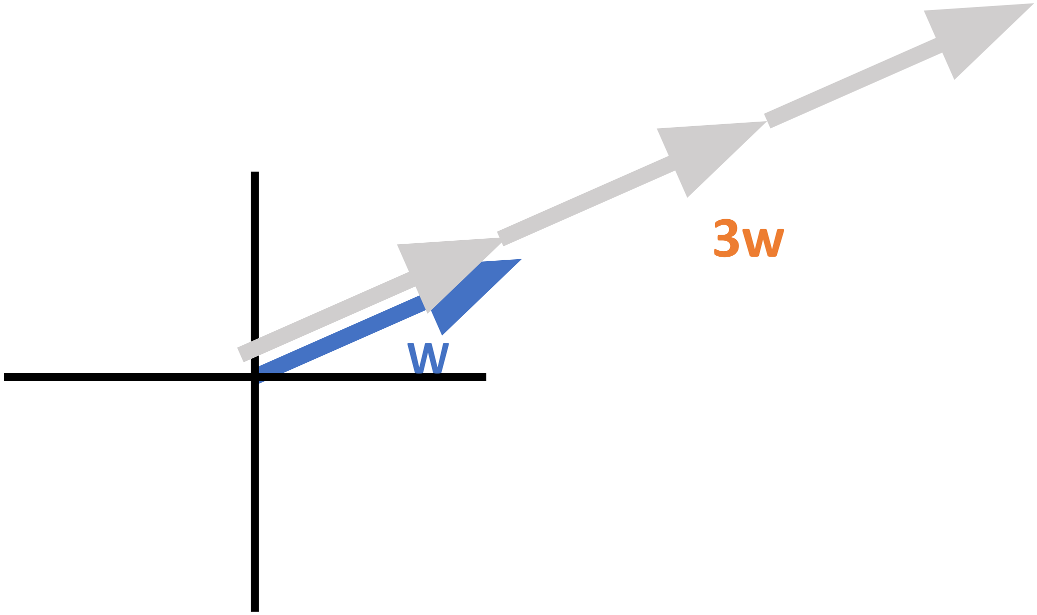

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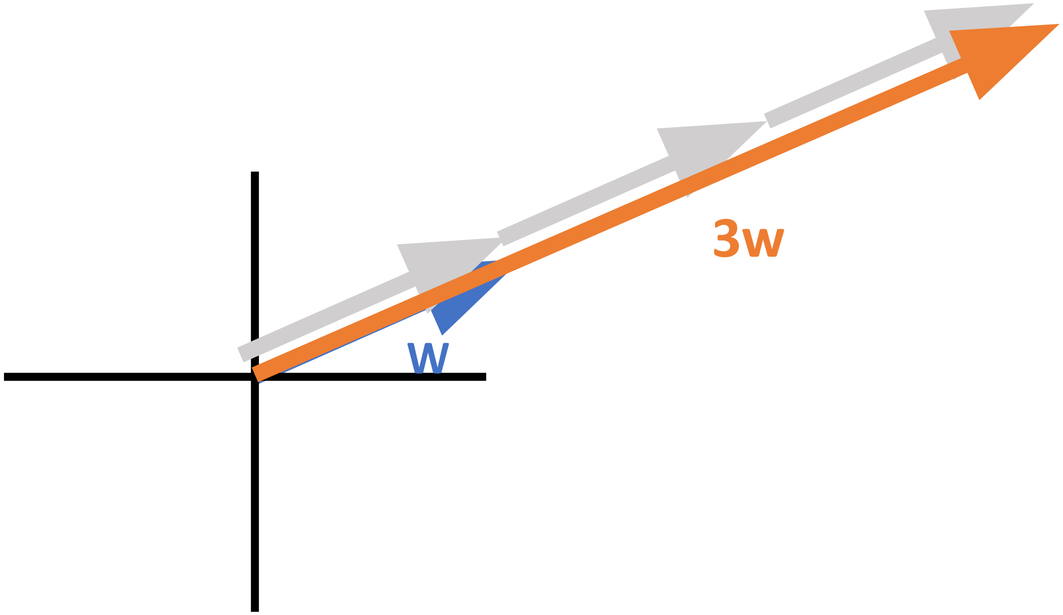


Scalar multiplication mean for our text dataset?

- Recall:
  - D1: [2, 3, 1, 1, 0]

- What does it mean to have document A = 2 * D1?

Next: Inner product (dot product)

- Definition $\boldsymbol{a} \cdot \boldsymbol{b} = \sum_i a_i b_i$

- E.g. 3-D vectors $\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^{3} a_i b_i$

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Inner product (dot product)

- This ends up being quite important in ML
- Corresponds to the weighted sum
- Many models make predictions using a weighted sum of the feature vector
- Example: price vector multiplied by quantity vector
- Makes a scalar: can be used as a measure of similarity (sometimes)

Inner product (dot product)

- Neat tricks with the inner product
- **One hot vector** times **feature vector** “**selects**” a particular element from the vector
- Example: a=[0, **1**, 0], b=[7, **5**, 8]

Inner product (dot product)

- Vectors can be squared
- E.g. b=[7, 5, 8], b^2 = ?

Length of a vector (Euclidean Norm)

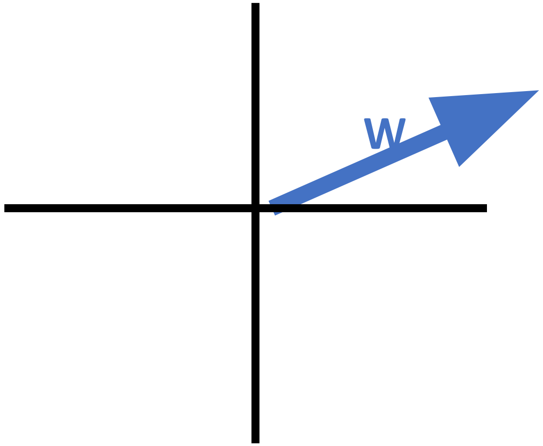
- Notation $\|a\|$

- Definition $$\begin{aligned}\|a\| &= \sqrt{a \cdot a} \\ &= \sqrt{\sum_i a_i^2}\end{aligned}$$

- You may have seen this in the Pythagorean theorem (length of the hypotenuse)

Vector Similarity

- It’s often useful to compute the similarity between two vectors



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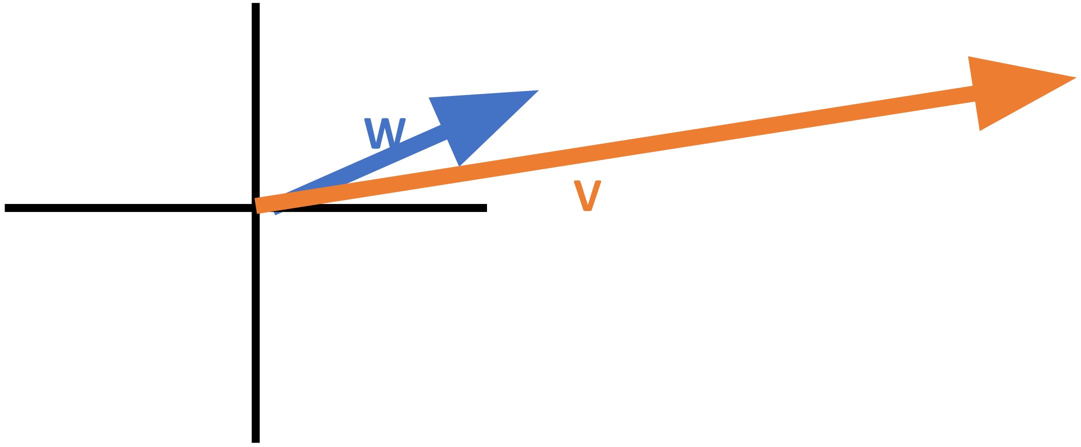
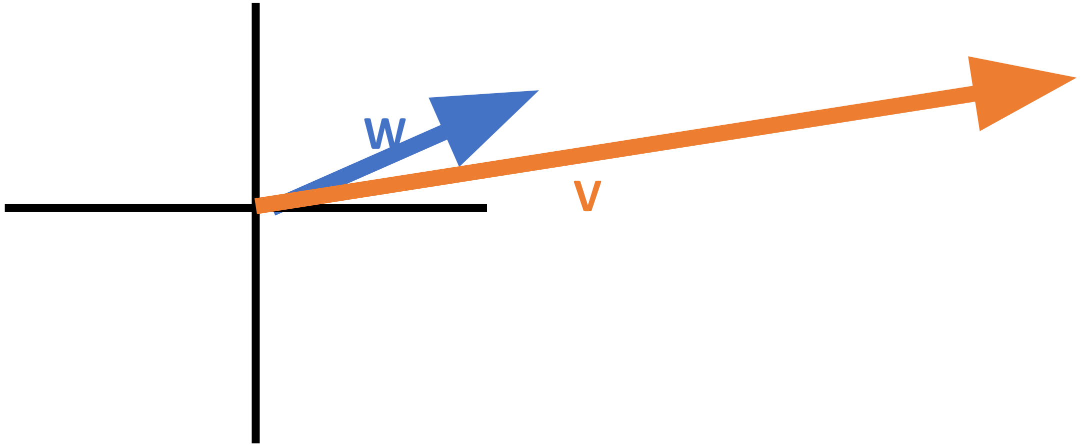


Vector Similarity

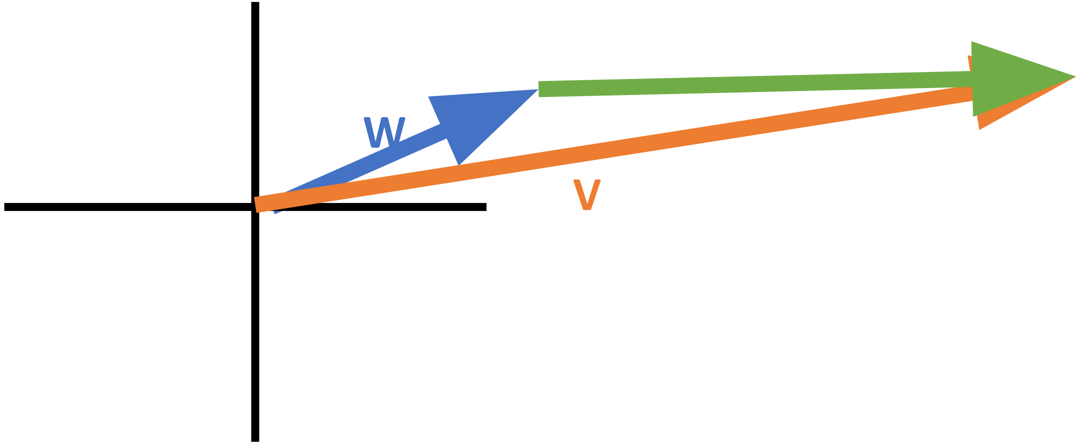
- Definition one: **Euclidean distance**



Vector Similarity

- Definition one: **Euclidean distance**

Distance between the tips/arrow end



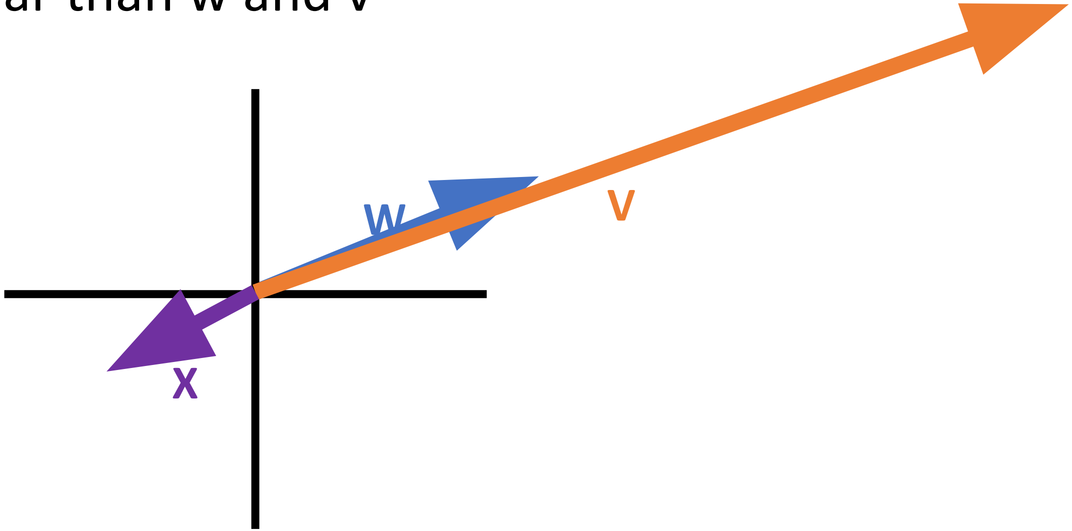
Vector similarity: Euclidean Distance

- Euclidean Distance

$$\sqrt{(w-v)^2}$$

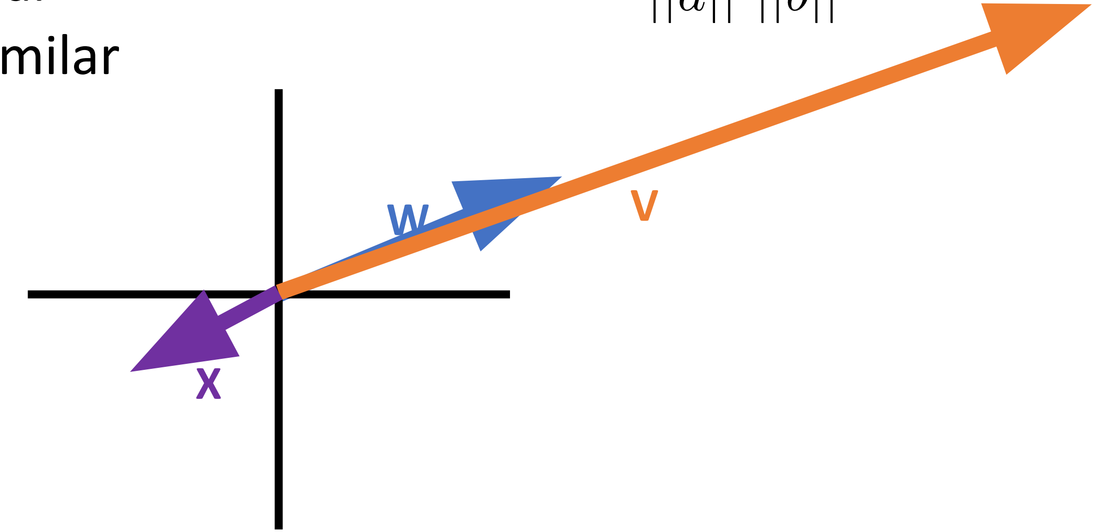
Vector similarity: Euclidean Distance

- Some problems with Euclidean Distance
- Here x and w are more similar than w and v
- Is that what we want?



Vector Similarity…Cosine Similarity

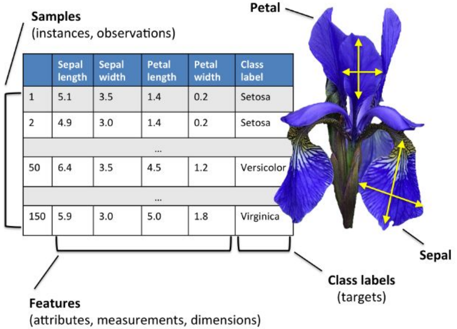
- Calculate the cosine of the angle between two vectors $\frac{a \cdot b}{||a||\ ||b||}$
  - Small angle -> very similar
  - Large angle -> very dissimilar
  - **Invariant to length, sensitive to direction**



Matrices

- Can be thought of as a function that transforms space

- In ML, our data is usually formatted into a **matrix**, where the **rows** correspond to **data samples**, and the **columns correspond to the features**



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Matrix Multiplication

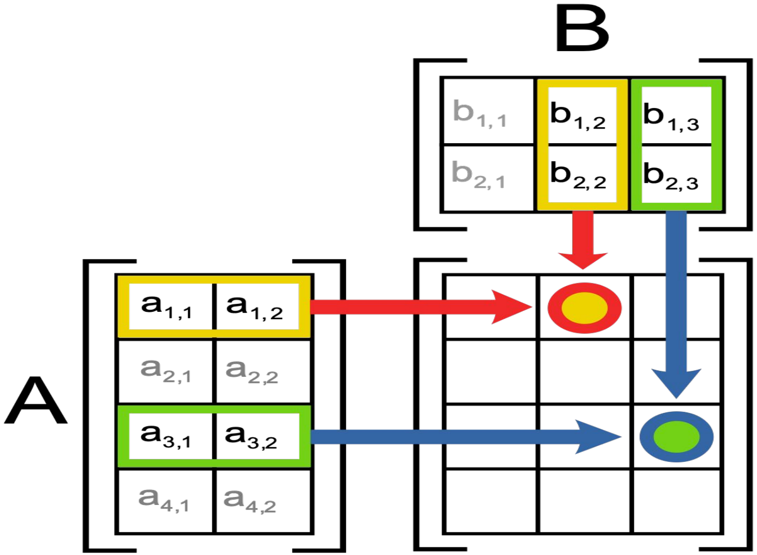
- E.g. A * B
- Each ***row*** vector of A dot product with each ***column*** vector of B
  - Again, **R**oll **C**all to remember which is rows and which is cols
- Scalar appears in resulting matrix where the row and column intersect
- The **# cols of A** must match **# of rows in B**

$$A \in \mathbb{R}^{n \times p}$$

$$B \in \mathbb{R}^{p \times m}$$

$$(AB) \in \mathbb{R}^{n \times m}$$

Matrix Multiplication



Special Matrices

- **Identity matrix** (often denoted I)
  - Square matrix, **All zeros**, **except** for the **diagonal elements** are **1**

$$\mathbf{I_n} = \underbrace{\left.\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}\right\}}_{n \text{ columns}} n \text{ rows}$$

- Called the Identity because IA = A, for all matrices A

Special Matrices and Invertible Matrices

- Inverse of a matrix $A(A^{-1}) = I$
- **Only** square matrices are invertible

- Finding the **inverse is complex for large matrices**
  - We won't worry about it, **the computer can do it** for us

- Some matrices are not invertible! (Singular) :(

Probability Overview

Many of these slides are derived from Alona Fyshe, Seyong Kim, Tom Mitchell, William Cohen, Eric Xing. Thanks!

Why do we care about probability?

- Helps us reason about how to make the best decision for cases were we need to generalize:

Temp	**Precip**	**Day**	**Clothes**	
22	None	Fri	Casual	**Walk**
3	None	Sun	Casual	**Walk**
10	Rain	Wed	Casual	**Walk**
30	None	Mon	Casual	**Drive**
20	None	Sat	Formal	**Drive**
25	None	Sat	Casual	**Drive**
-5	Snow	Mon	Casual	**Drive**
27	None	Tue	Casual	**Drive**
24	Rain	Mon	Casual	**?**

Recall: Generalization

- Dealing with previously unseen cases
- Will she walk or drive?

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27	None	Tue	Casual	**Drive**
24	Rain	Mon	Casual	**?**

We might plausibly make any of the following arguments:

- She's going to walk because it's raining today and the only other time it rained, she walked.
- She's going to drive because she has always driven on Mondays…

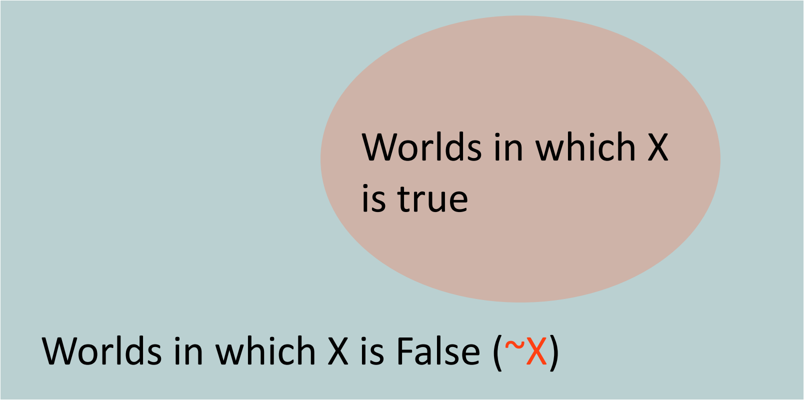
Terminology: Random Variables

- Informally, X is **a random variable** if
  - X denotes something about which we are uncertain
  - perhaps the outcome of a randomized experiment
    - e.g. rolling a die

- Examples
  - X = The hometown of a randomly drawn person from our class
    - multivalued
  - X = True if two randomly drawn persons from our class have same birthday
    - binary

Functions of Random Variables

- Define P(X) as “the fraction of possible worlds in which X is true” or “the fraction of times X holds, in repeated runs of the random experiment”

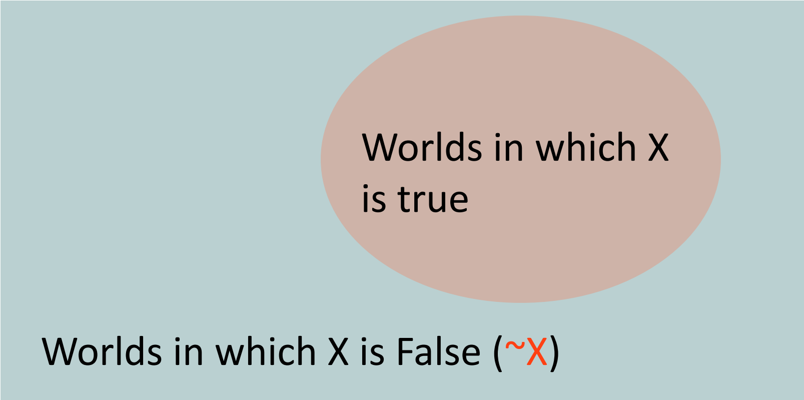


Functions of Random Variables

- Define P(X) as "the fraction of possible worlds in which X is true" or "the fraction of times X holds, in repeated runs of the random experiment"
  - the set of possible worlds is called the sample space, S

Blue Rectangle:
Sample space of all possible worlds (S)

Area = 1 (all possible things)



Worlds in which X is true

Worlds in which X is False (~X)

P(X) = Area of reddish oval

$0 < P(X) < 1$

A little formalism

More formally, we have

- a **sample space S** (e.g., set of students in our class)
  - aka the set of possible worlds
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  - e.g., the subset of S for which (handedness=r) AND (eyeColor=blue)

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  - e.g., the subset of S for which handedness = r
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- We are often interested in **probabilities of specific events** and **of specific events conditioned on other specific events**

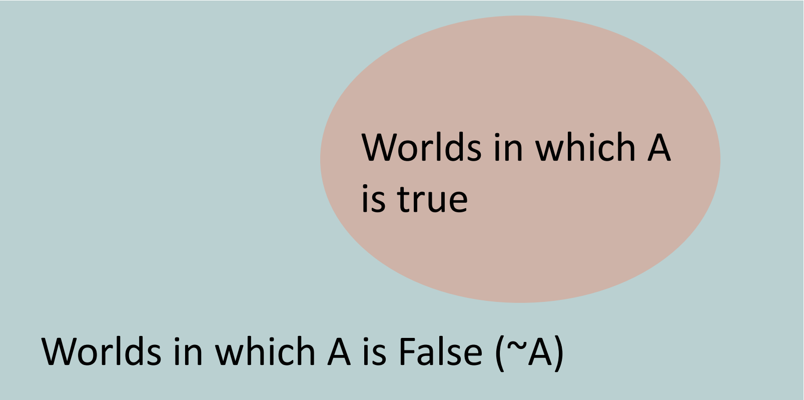
The Axiom(s) of Probability

- Assume binary random variables A and B.

The Axiom(s) of Probability

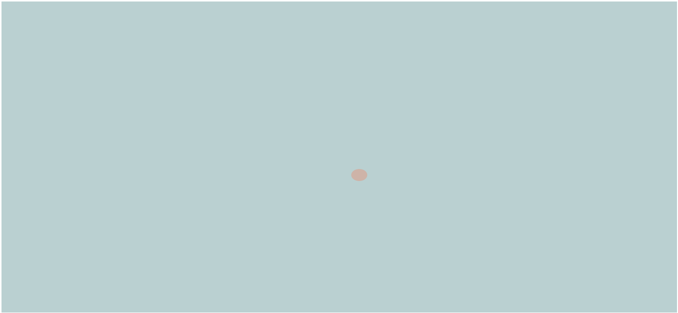
- Assume binary random variables A and B.
  - $0 <= P(A) <= 1$
  - P(True) = 1
  - P(False) = 0
  - **P(A or B) = P(A) + P(B) – P(A and B)**

Visualizing Probability Axioms



Towards Interpreting the axioms

- **P(A) = 0**



The area of A can’t get any smaller than 0

And a zero area would mean no world could ever have A true

P(True) = 0

Towards Interpreting the axioms

- **P(A) = 1**

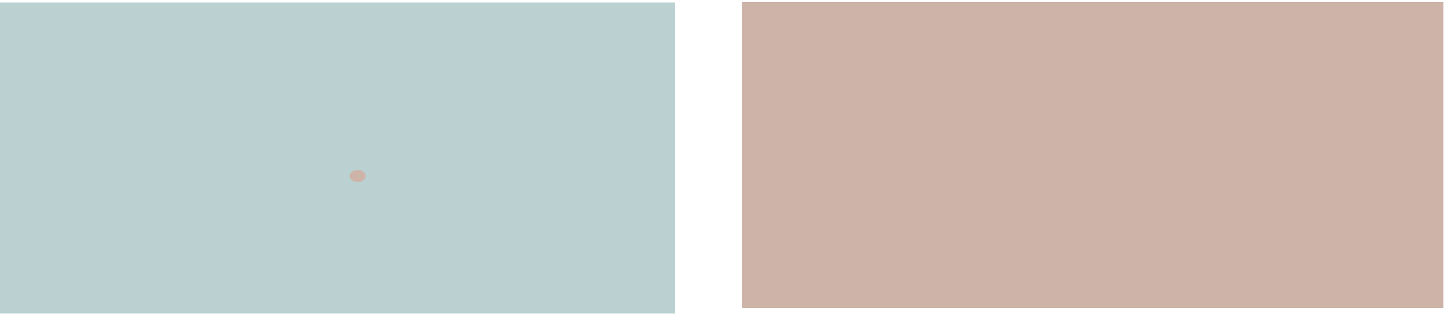
The area of A can’t get any bigger than 1

And an area of 1 would mean all worlds will have A true

P(True) = 1

Towards Interpreting the Axioms

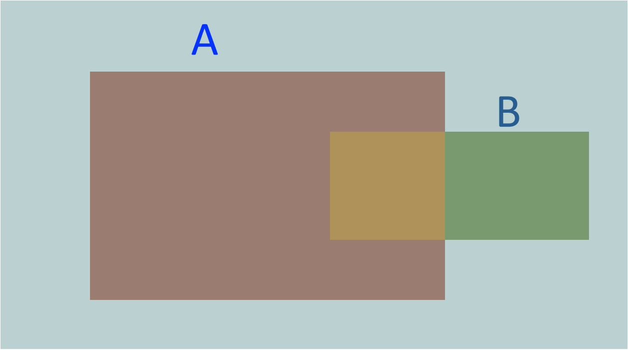
$$0 <= P(A) <= 1$$



Towards Interpreting the axioms

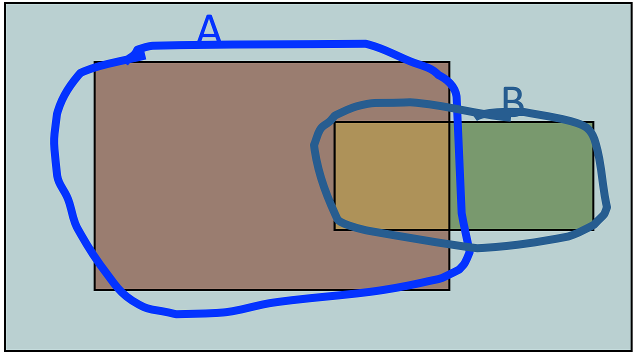
- P(A or B) = P(A) + P(B)

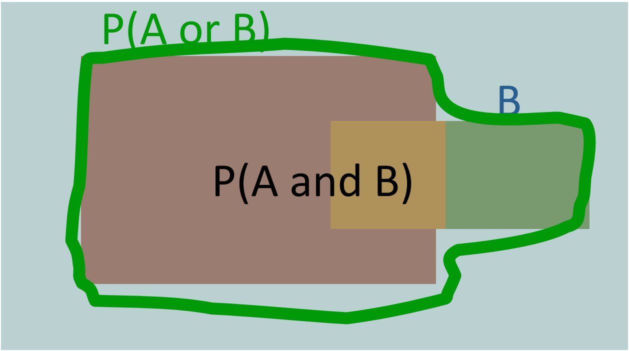
[WRONG! but why?]



Towards Interpreting the axioms

- **P(A or B) = P(A) + P(B) - P(A and B)**





Simple addition and subtraction

Another useful theorem

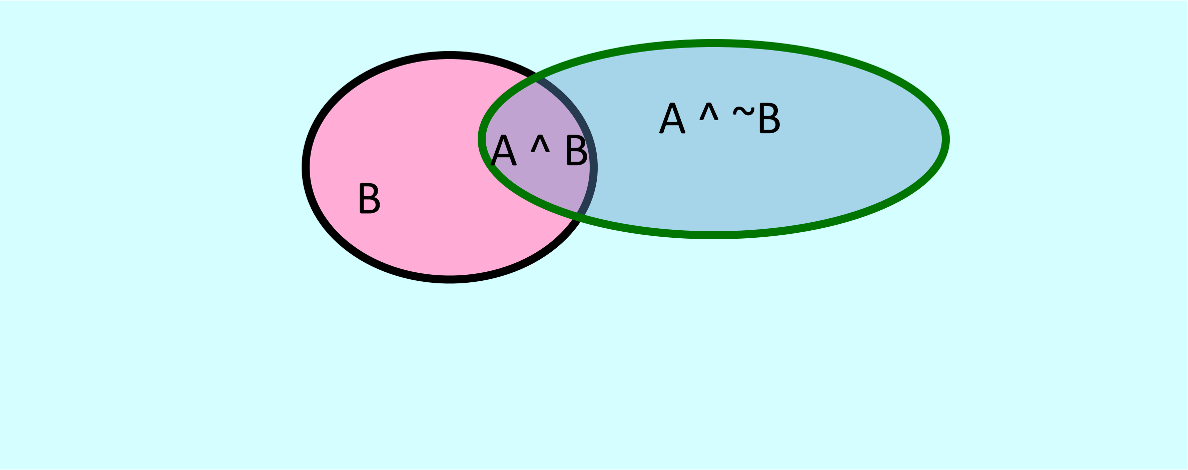
0 <= P(A) <= 1, P(True) = 1, P(False) = 0,

P(A or B) = P(A) + P(B) - P(A and B)

□ P(A) = P(A ^ B) + P(A ^ ~B)

Elementary Probability in Pictures

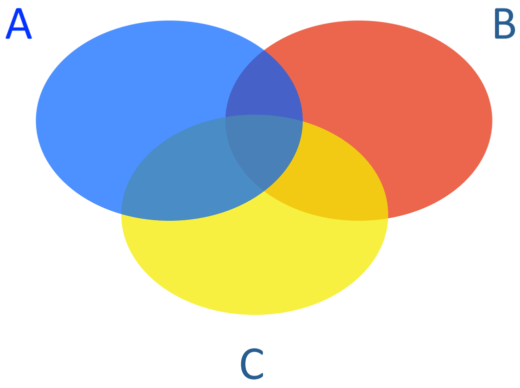
- P(A) = P(A ^ B) + P(A ^ ~B)



- P(A or B) = P(A ^ B) + P(A ^ ~B) + P(~A ^ B)

Extending the Axiom

- P(A or B or C) = ?



Multivalued Discrete Random Variables

- Suppose A can take on more than 2 values
- A is a random variable with arity k if it can take on exactly one value out of $\{v_1, v_2, \ldots v_k\}$
- *Example:* A=*{1,2,3….,20}:* good for 20-sided dice games
- Notation: let’s write the event AHasValueOf*v* as “A=*v*”

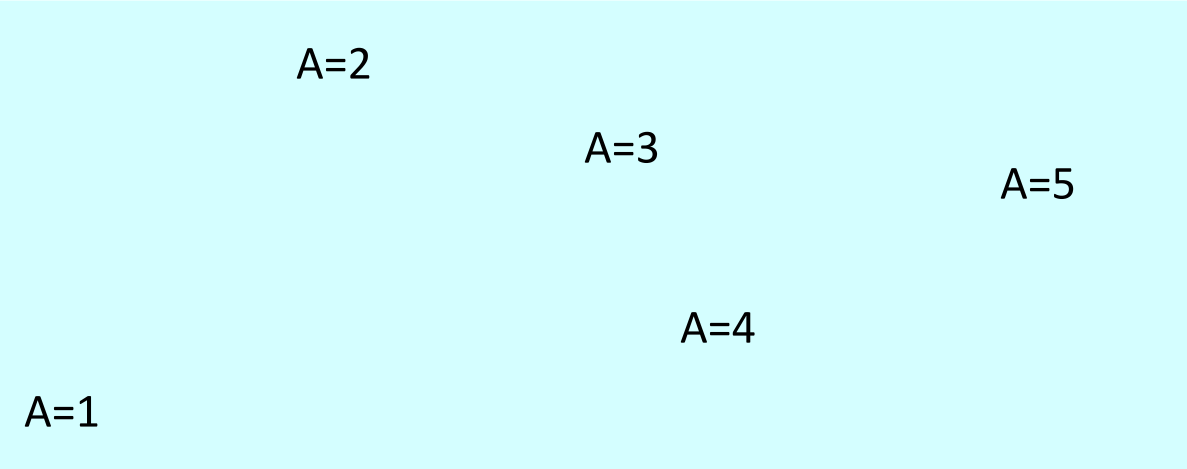
$$P(A = v_i \wedge A = v_j) = 0 \text{ if } i \neq j$$

- Thus… $P(A = v_1 \vee A = v_2 \vee \ \ldots \ \vee A = v_k)$

Elementary Probability in Pictures

$$\sum_{j=1}^{k} P(A = v_j) = 1$$

(Law of total probability)



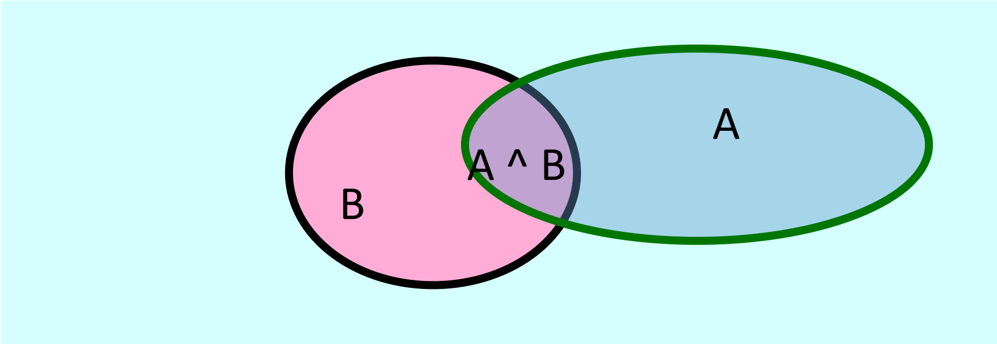
Bunny Break

Definition of Conditional Probability

$$P(A|B) = \frac{P(A \wedge B)}{P(B)}$$

We say “probability of A given b”

Foundation for Bayes’ Rule!



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Corollary: The Chain Rule

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$$P(A \wedge B \wedge C) = P(A|B \wedge C)\ P(B \wedge C)$$

$$=$$

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$$= P(A|B \wedge C)\ P(B|C)\ P(C)$$

Independent Events

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  P(A and B)=P(A)*P(B)
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$$(\text{if}) = P(A)P(B)$$

$$- > P(A|B) = P(A)$$

Independent Events

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  P(A and B)=P(A)*P(B)

- Intuition: knowing A tells us nothing about the value of B (and vice versa)
- From chain rule

$$P(A \wedge B) = P(A|B)\, P(B) = P(A)P(B)$$

$$\rightarrow P(A|B) = P(A)$$

- **You frequently need to assume the independence of *something* to solve a learning problem.**

Continuous Random Variables

- The discrete case: sum over all values of A is 1

$$\sum_{j=1}^{k} P(A = v_j) = 1$$

- The continuous case: infinitely many values for A and the *integral* is 1

$$\int_{-\infty}^{\infty} f_P(x)\,dx = 1$$

f(x) is a probability density function (pdf)

also

$$\forall x,\ f_P(x) \geq 0$$

….

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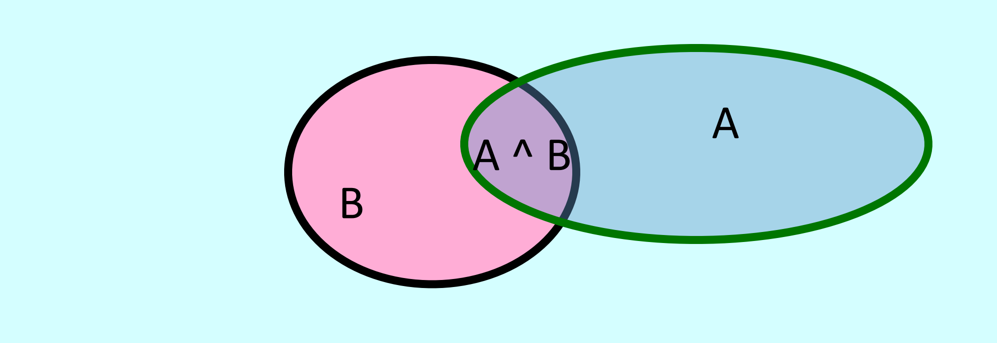
Bayes Rule

Let’s write two expressions for P(A ^ B)



$$P(A \wedge B) = P(A|B)\,P(B)$$

$$P(A \wedge B) = P(B|A)P(A)$$

Bayes Rule

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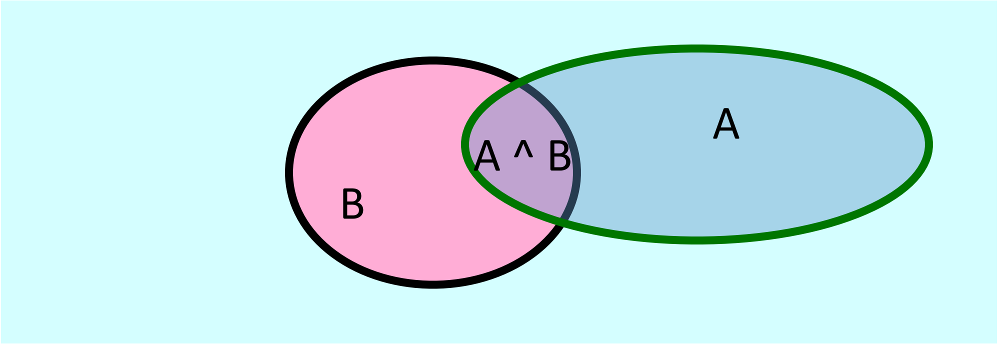

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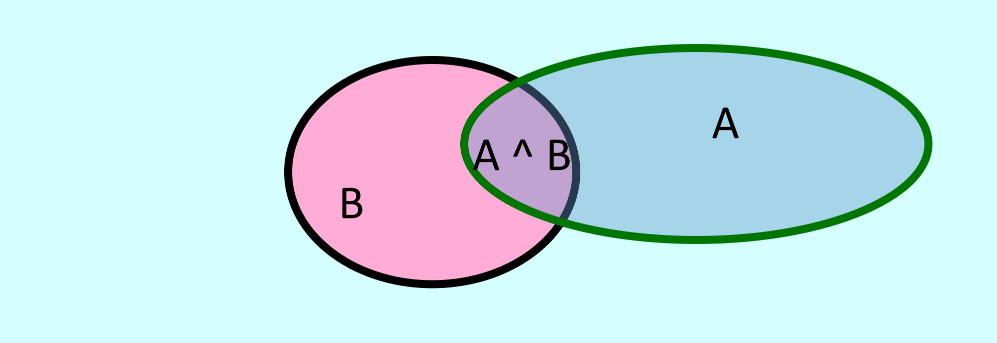

$$P(A \wedge B) = P(A|B)\ P(B)$$

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$$P(A|B)\ \mathbf{P(B)} = P(B|A)P(A)$$

$$P(A|B) = \frac{P(B|A) * P(A)}{P(B)}$$ Bayes’ rule

we call P(A) the “prior”

and P(A|B) the “posterior”

Bayes, Thomas (1763) An essay towards solving a problem in the doctrine of chances. *Philosophical Transactions of the Royal Society of London,* **53:370-418**

…by no means merely a curious speculation in the doctrine of chances, but necessary to be solved in order to a sure foundation for all our reasonings concerning past facts, and what is likely to be hereafter…. necessary to be considered by any that would give a clear account of the strength of *analogical* or *inductive reasoning*…

Other Forms of Bayes Rule

$$P(A|B) = \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid \sim A)P(\sim A)}$$

$$P(A|B \wedge X) = \frac{P(B \mid A \wedge X)P(A \wedge X)}{P(B \wedge X)}$$

Recall useful theorem Slide 72 P(B) = P(B ^ A) + P(B ^ ~A) , and same as before just different letters P(C ^ D) = P(C|D) P(D)

Applying Bayes Rule

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\sim A)P(\sim A)}$$

A = you have the flu,   B = you just coughed

Assume:

P(A) = 0.05

Applying Bayes Rule

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\sim A)P(\sim A)}$$

A = you have the flu,   B = you just coughed

Assume:
P(A) = 0.05

Also assume the following information is known to you
P(B|A) = 0.80
P(B| ~A) = 0.4

what is P(flu | cough) = P(A|B)?

Next! Joint distribution

- Probability of >1 thing happening at the same time
  - Probability it will rain today and I forgot my umbrella
    - P(rain=true,umbrella=false)

The Joint Distribution

Example: Boolean variables A, B, C

Recipe for making a joint distribution of M variables:

The Joint Distribution

Example: Boolean variables A, B, C

Recipe for making a joint distribution of M variables:

1. Make a truth table listing all combinations of values of your variables (if there are **M Boolean variables** then the table will have **2^M** rows).

A	**B**	**C**
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

The Joint Distribution

Example: Boolean variables A, B, C

Recipe for making a joint distribution of M variables:

1. Make a truth table listing all combinations of values of your variables (if there are M Boolean variables then the table will have 2^M rows).
2. For **each combination** of values, say **how probable** it is.

A	**B**	**C**	**Prob**
0	0	0	0.30
0	0	1	0.05
0	1	0	0.10
0	1	1	0.05
1	0	0	0.05
1	0	1	0.10
1	1	0	0.25
1	1	1	0.10

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Recipe for making a joint distribution of M variables:

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2. For each combination of values, say how probable it is.
3. If you **subscribe to the axioms of probability**, those numbers **must sum to 1**.

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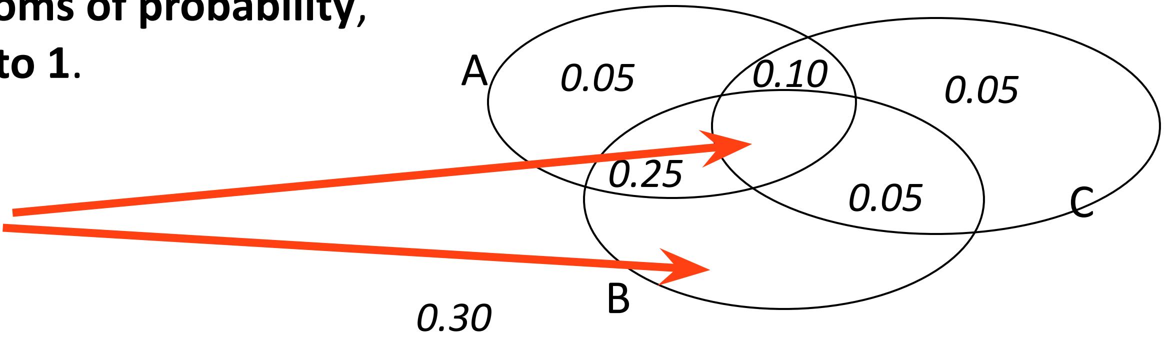
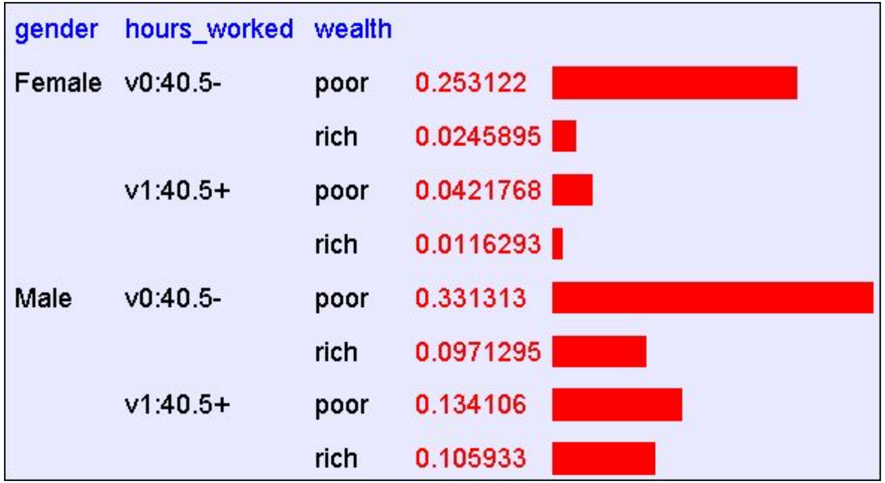


What goes here?

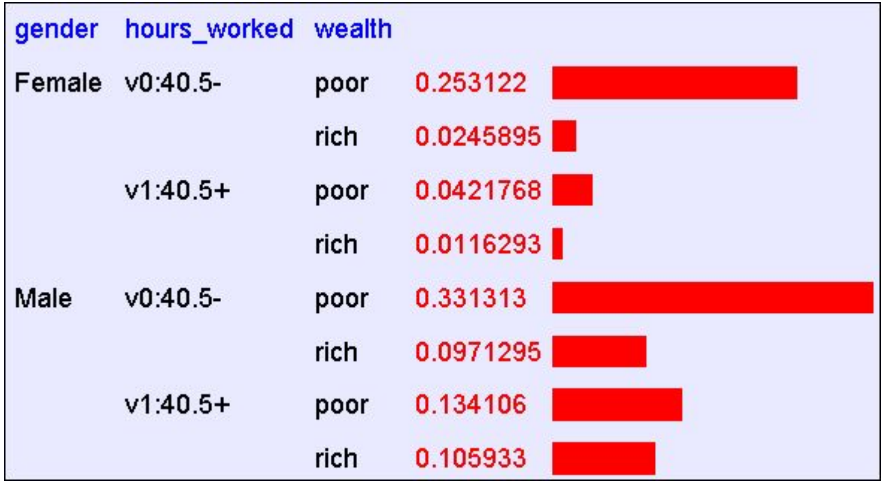
Joint Probability Distribution



gender	hours_worked	wealth	
Female	v0:40.5-	poor	0.253122
		rich	0.0245895
	v1:40.5+	poor	0.0421768
		rich	0.0116293
Male	v0:40.5-	poor	0.331313
		rich	0.0971295
	v1:40.5+	poor	0.134106
		rich	0.105933

Once you have the joint distribution, you can **ask for the probability** of **any logical expression** involving your attribute

Using the Joint Distribution



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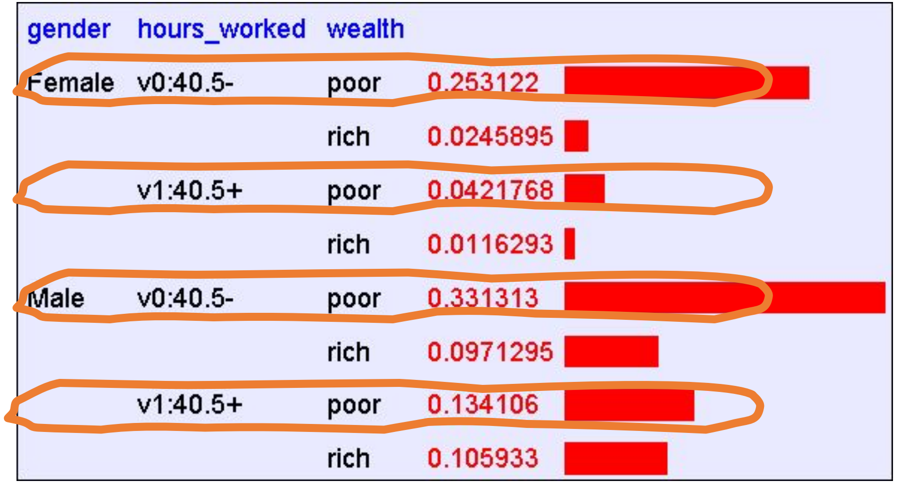
P(Poor) = 0.7604

$$P(E) = \sum_{\text{rows matching } E} P(\text{row})$$

Using the Joint Distribution



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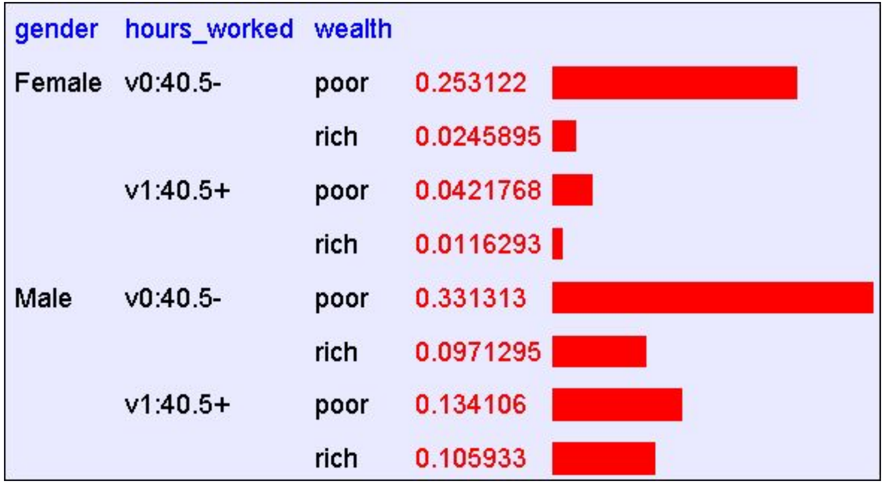
Inference with the Joint



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$$P(E_1 \mid E_2) = \frac{P(E_1 \wedge E_2)}{P(E_2)} = \frac{\sum\limits_{\text{rows matching } E_1 \text{ and } E_2} P(\text{row})}{\sum\limits_{\text{rows matching } E_2} P(\text{row})}$$

Next! Maximum Likelihood Estimation (MLE)

Rich vs Poor

What is the probability of a person being rich, given you know nothing else about that person?

3:2

Let’s say 3/5?

We assume that the wealth of the people in our dataset *D* is independently distributed

θ = Probability of being rich = P(rich)

? = Probability of being poor = P(poor)

Let’s say 3/5?

We assume that the wealth of the people in our dataset *D* is independently distributed

θ = Probability of being rich = P(rich)

? = Probability of being poor = P(poor)

D = { r, p, r, r, p} α_r = # rich α_p= # poor

$$P(D) = P(r \text{ and } p \text{ and } r \text{ and } r \text{ and } p)$$

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We assume that the wealth of the people in our dataset *D* is independently distributed

θ = Probability of being rich = P(rich)

? = Probability of being poor = P(poor)

D = { r, p, r, r, p} α_r = # rich α_p= # poor

$$
\begin{aligned}
P(D) =& P(r \text{ and } p \text{ and } r \text{ and } r \text{ and } p) \\
=& P(rich) * P(poor) * P(rich) * \\
& P(rich) * P(poor)
\end{aligned}
$$

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=& P(rich) * P(poor) * P(rich) * \\
& \quad P(rich) * P(poor) \\
=& \theta * (1-\theta) * \theta * \theta * (1-\theta) \\
=& (1-\theta)^{\alpha_p} * \theta^{\alpha_r}
\end{aligned}
$$

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We assume that the wealth of the people in our dataset *D* is independently distributed

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$$P(D) = P(r \text{ and } p \text{ and } r \text{ and } r \text{ and } p)$$

$$= P(rich) * P(poor) * P(rich) *$$

$$P(rich) * P(poor)$$

$$= \theta * (1-\theta) * \theta * \theta * (1-\theta)$$

$$= (1-\theta)^{\alpha_p} * \theta^{\alpha_r}$$

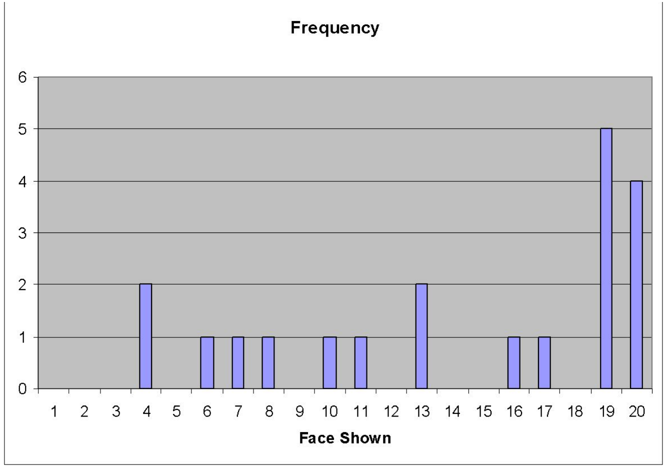
$$\boxed{\underset{\theta}{\text{argmax}}\ P(D) = (1-\theta)^{\alpha_F} * \theta^{\alpha_H}}$$

That’s Maximum Likelihood Estimation (MLE)

It’s not always the best solution…

Issues with MLE estimate

I bought a loaded 20-faced die (d20) on EBay…but it didn't come with any specs. How can I find out how it behaves?



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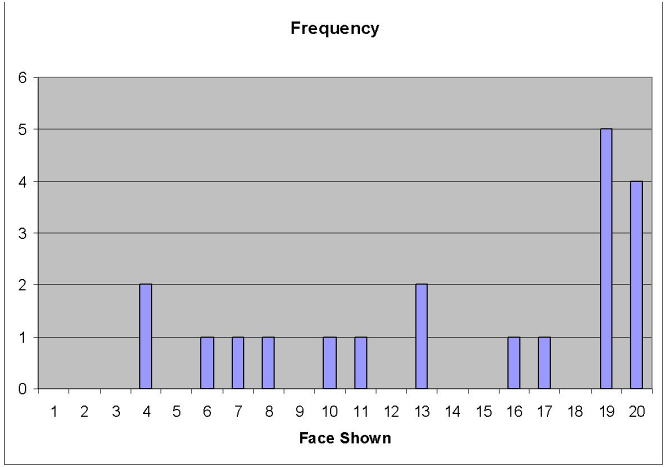

1. Collect some data (20 rolls)

Issues with MLE estimate

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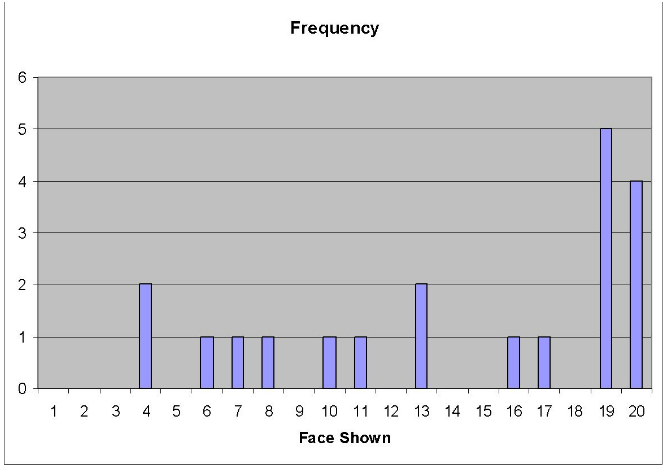

1. Collect some data (20 rolls)
2. Estimate P(i)=CountOf(rolls of i)/CountOf(any roll)

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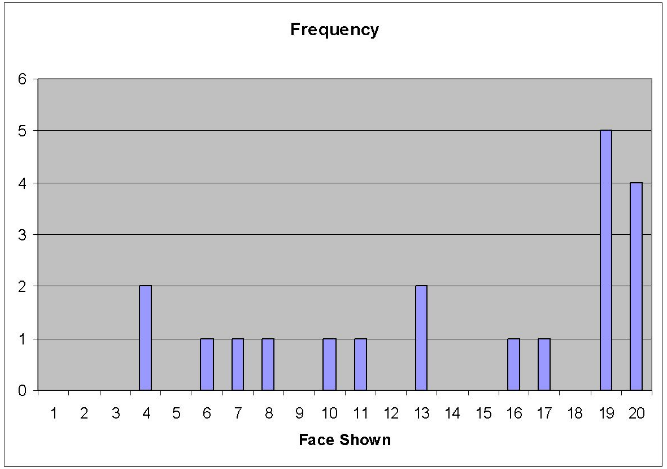


P(1)=0

P(2)=0

P(3)=0

P(4)=0.1

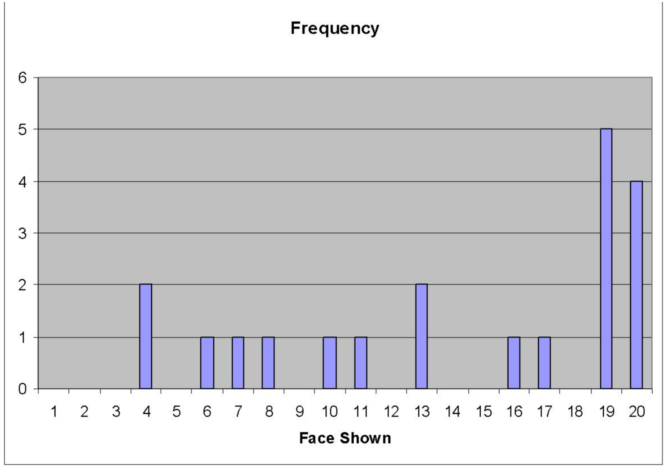
…

P(19)=0.25

P(20)=0.2

Issues with MLE estimate

I bought a loaded 20-faced die (d20) on EBay…but it didn't come with any specs. How can I find out how it behaves?



P(1)=0

P(2)=0

P(3)=0

P(4)=0.1

…

P(19)=0.25

P(20)=0.2

But: Do I really think it's *impossible* to roll a 1,2 or 3?

A better solution?

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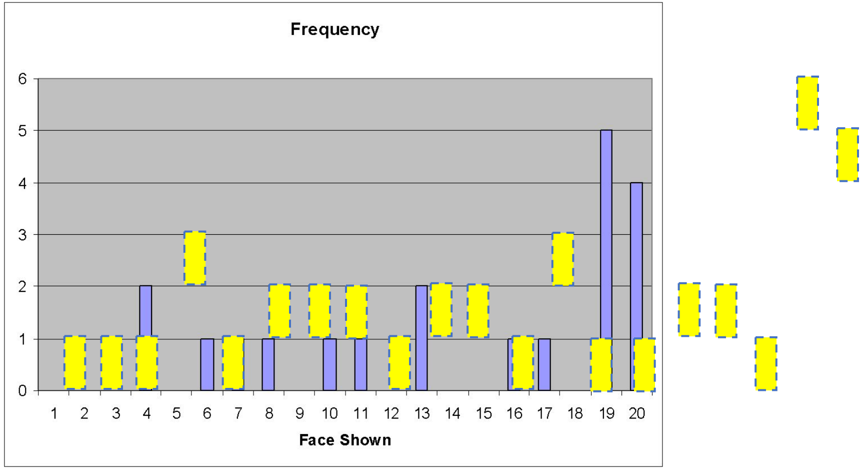

1. Collect some data (20 rolls)
2. Estimate P(i)

A better solution

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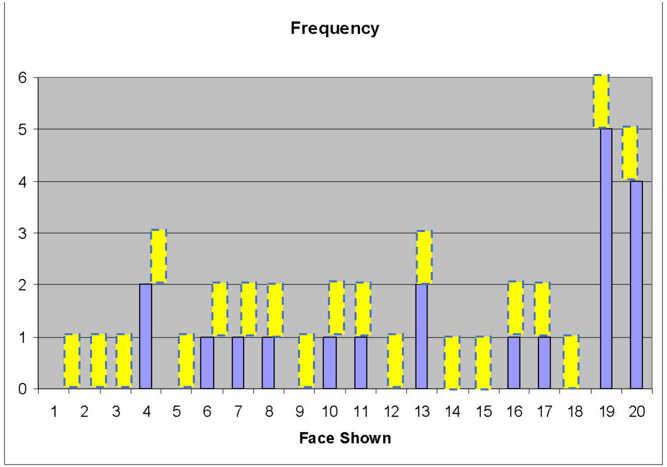
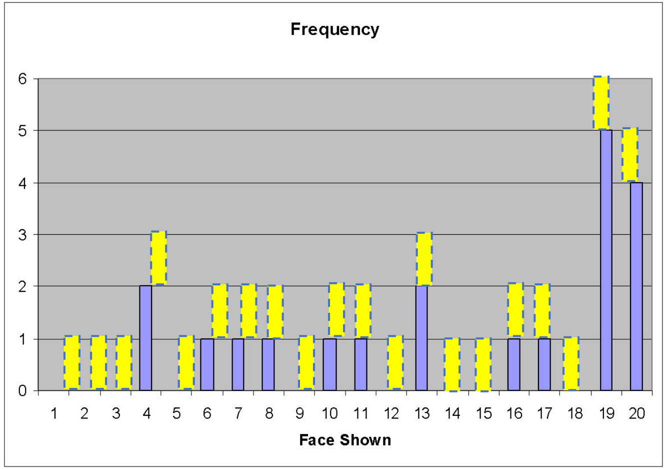


0. *Imagine* some data (20 rolls, each i shows up 1x)
1. Collect some data (20 rolls)
2. Estimate P(i)

A better solution?



P(1)=1/40

P(2)=1/40

P(3)=1/40

P(4)=(2+1)/40

…

P(19)=(5+1)/40

P(20)=(4+1)/40=1/8

$$\hat{P}(i) = \frac{CountOf(i)+1}{CountOf(ANY)+CountOf(IMAGINED)}$$

0.2 *vs.* 0.125 – really different! Maybe I should "imagine" less data?

What if we know that poor people are much more common than rich people?

We have a belief about θ

- P(θ|D) = P(D|θ)*P(θ)/P(D)

Now we can incorporate our belief about θ

We have a belief about θ

- $P(\theta|D) = P(D|\theta)*P(\theta)/P(D)$

$$\propto$$

Now we can incorporate our belief about θ

We have a belief about θ

- $P(\theta|D) = P(D|\theta)*P(\theta)/P(D)$

$$\propto P(D|\theta)*P(\theta)$$

Now we can incorporate our belief about θ

This is a MAP (Maximum A Posteriori) Estimate

Conjugate Prior

- Our likelihood so far has been based on a Bernoulli distribution.
- **Beta is a conjugate prior to Bernoulli**
  - This means their pdfs (probability density functions) play nice together
  - **P(D|θ)*P(θ)** will be easy to deal with
  - Called the posterior likelihood

Estimating Parameters

- Maximum Likelihood Estimate (MLE): choose θ that maximizes probability of observed data

$$\widehat{\theta} = \arg\max_{\theta} \; P(\mathcal{D} \mid \theta)$$

- Maximum a Posteriori (MAP) estimate: choose θ that is most probable given **prior probability and the data**

$$\widehat{\theta} = \arg\max_{\theta} \; P(\theta \mid \mathcal{D})$$

$$= \arg\max_{\theta} \; = \frac{P(\mathcal{D} \mid \theta) P(\theta)}{P(\mathcal{D})}$$

A tutorial:
http://www.cs.ubc.ca/~murphyk/Teaching/CS340-Fall06/reading/bernoulli.pdf

Thanks, see you Tuesday!